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Generalized Uncertainty Principle from Quantum Geometry

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The generalized uncertainty principle of string theory is derived in the framework of quantum geometry by taking into account the existence of an upper limit on the acceleration of massive particles.

1. INTRODUCTION

The problem of reconciling quantum mechanics (QM) with general relativity is a task of modern theoretical physics which has not yet found a consistent and satisfactory solution. The difficulty arises because general relativity deals with events which define the world-lines of particles, while QM does not allow the definition of trajectory; in fact the determination of the position of a quantum particle involves a measurement which introduces an uncertainty into its momentum (Wigner, 1957; Saleker and Wigner, 1958; Feynman and Hibbs, 1965).

These conceptual difficulties have their origin, as argued in Candelas and Sciama (1983) and Donoghue *et al.* (1984, 1985), in the violation, at the quantum level, of the weak principle of equivalence on which general relativity is based. Such a problem becomes more involved in the formulation of a quantum theory of gravity owing to the nonrenormalizability of general relativity when one quantizes it as a local quantum field theory (QFT) (Birrel and Davies, 1982).

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Nevertheless, one of the most interesting consequences of this unification is that in quantum gravity there exists a minimal observable distance on the order of the Planck distance, $l_P = \sqrt{G\hbar/c^3} \sim 10^{-33}$ cm, where G is the Newton constant. The existence of such a fundamental length is a dynamical phenomenon due to the fact that, at Planck scales, there are *fluctuations* of the background metric, *i.e.*, a limit of the order of the Planck length appears when quantum fluctuations of the gravitational field are taken into account.

In the absence of a theory of quantum gravity, one tries to analyze quantum aspects of gravity retaining the gravitational field as a classical background, described by general relativity, and interacting with a matter field. This *semiclassical approximation* leads to QFT and QM in curved space-time and may be considered as a preliminary step toward a complete quantum theory of gravity. In other words, we take into account a theory where geometry is classically defined while the source of the Einstein equations is an effective stress-energy tensor where contributions of matter quantum fields, gravity self-interactions, and quantum matter–gravity interactions appear (Birrel and Davies, 1982).

The canonical commutation relations between the momentum operator p^ν and position operator x^μ, which in Minkowski space-time are $[x^\mu, p^\nu] = i\hbar\eta^{\mu\nu}$, in a curved space-time with metric $g_{\mu\nu}$ can be generalized as

$$[x^\mu, p^\nu] = i\hbar g^{\mu\nu}(x) \tag{1}$$

Equation (1) contains the gravitational effects of a particle in first quantization scheme. Its validity is confined to asymptotically flat curved space-time so that the tensor metric can be decomposed as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is the (local) perturbation to the flat background (Ashtekar, 1990). We note that the usual commutation relations between position and momentum operators in Minkowski space-time are obtained by using the vierbein formalism, i.e., by projecting the commutator and the metric tensor on the tangent space.

As is well known, a theory containing a fundamental length on the order of l_P (which can be related to the extension of particles) is string theory. It provides a consistent theory of quantum gravity and avoids the above-mentioned difficulties. In fact, unlike point particle theories, the existence of a fundamental length plays the role of a natural cutoff. In such a way the ultraviolet divergences are avoided without appealing to renormalization and regularization schemes (Green *et al.*, 1987).

By studying string collisions at Planckian energies and through a renormalization group-type analysis (Veneziano, 1986; Amati *et al.*, 1987, 1988, 1989, 1990; Gross and Mende, 1987, 1988; Konishi *et al.*, 1990; Guida and Konishi, 1991; Yonega, 1989), the emergence of a minimal observable distance yields the generalized uncertainty principle

$$\Delta x \geq \frac{\hbar}{2\Delta p} + \frac{\alpha}{c^3} G \, \Delta p \tag{2}$$

Here, α is a constant. At energies much below the Planck mass, $m_P = \sqrt{\hbar c/G} \sim 10^{19}$ GeV/c^2, the extra term in Eq. (2) is irrelevant, and the Heisenberg relation is recovered, while as we approach the Planck energy this term becomes relevant and is related to the minimal observable length.

The purpose of this paper is to recover the generalized uncertainty principle, (2), in the framework of quantum geometry theory (Caianiello, 1979, 1980a, b, 1992). It incorporates quantum aspects into space-time geometry so that one-particle QM acquires a geometric interpretation. Its formulation is based on the fact that the position and momentum operators are represented as covariant derivatives with an appropriate connection in the eight-dimensional manifold and the quantization is geometrically interpreted as the curvature of phase space.

A consequence of this geometric approach is the existence of a maximal acceleration defined as the upper limit to the proper acceleration $\mathcal{A}$ experienced by massive particles along their worldlines (Caianiello, 1981, 1984; Caianello *et al.*, 1982). It can be interpreted as being mass-dependent, $\mathcal{A}_m = 2mc^3/\hbar$ (m is the mass of particle), or as an universal constant, $\mathcal{A} = m_P \, c^3/\hbar$ (m_P is the Planck mass). Since the regime of validity of (2) is at Planck scales, in order to derive it from quantum geometry, we will consider the maximal acceleration as depending on the Planck mass.

The existence of a maximal acceleration has several implications for relativistic kinematics (Scarpetta, 1984), an energy spectrum of a uniformly accelerated particle (Caianiello, 1990a), a Schwarzschild horizon (Gasperini and Scarpetta, 1989), expansion of the very early universe (Caianiello *et al.*, 1991), tunneling from *nothing* (Capozziello and Feoli, 1993; Caianiello, *et al.*, 1994), and mass of the Higgs boson (Kuwata, 1996). It also makes the metric observer-dependent, as conjectured by Gibbons and Hawking (1977) and leads, in a natural way, to hadronic confinement (Caianiello *et al.*, 1988). The regularizing properties of the maximal acceleration have been recently analyzed in Feoli *et al.* (1999), and its applications in the framework of string theory have been studied in Feoli (1993) and McGuigan (1994).

Moreover, concrete experimental tests of the consequence of a maximal acceleration have been proposed in Caianiello *et al.* (1990a, b), Papini *et al.* (1995), and Lambiase *et al.* (1998).

Limiting values for the acceleration were also derived by several authors on different grounds and applied to many branches of physics (Brandt, 1983, 1984, 1989; Das, 1980; Frolov and Sanchez, 1991; Papini, 1992, 1995; Pati, 1992; Sanchez, 1993; Toller, 1988, 1990, 1991; Vigier, 1991, Wood, *et al.*, 1989; Wood and Papini, 1992).

The paper is organized as follows. In Section 2, we briefly discuss quantum geometry, recalling only the main topics used in this paper. Section 3 derives the generalized uncertainty principle from quantum geometry. Conclusions are discussed in Section 4.

2. QUANTUM GEOMETRY

Quantum geometry includes the effects of the maximal acceleration on the dynamics of particles in enlarging the space-time manifold to an eight-dimensional space-time tangent bundle TM, i.e., $M_8 = V_4 \otimes TV_4$, where V_4 is the background space-time equipped with metric $g_{\mu\nu}$. In this way, the invariant line element defined in M_8 is generalized as

$$d\tilde{s}^2 = g_{AB}\, dX^A\, dX^B, \qquad A, B = 1, \ldots, 8 \tag{3}$$

where the coordinates of M_8 are

$$X^A = \left(x^\mu; \frac{c^2}{\mathcal{A}} \frac{dx^\mu}{ds}\right), \qquad \mu = 1, \ldots, 4 \tag{4}$$

ds is the usual infinitesimal element line, $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$, $\mathcal{A}$ is the maximal acceleration, and

$$g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu} \tag{5}$$

From Eq. (5), it follows that the generalized line element (3) can be written as

$$d\tilde{s}^2 = g_{\mu\nu}\left(dx^\mu\, dx^\nu + \frac{c^4}{\mathcal{A}^2}\, d\dot{x}^\mu\, dx^\nu\right) \tag{6}$$

An embedding procedure can be developed (Caianiello *et al.*, 1990b) in order to find the effective space-time geometry where a particle moves when the constraint of the maximal acceleration is present. In fact, if we find the parametric equations that relate the velocity field $\dot{x}^\mu$ to the first four coordinates x^μ, we can calculate the effective four-dimensional metric $\tilde{g}_{\mu\nu}$ induced on the hypersurface locally embedded in M_8. For a particle of mass m accelerating along its worldline, Eq. (6) implies that it behaves dynamically as if it is embedded in a space-time with the metric

$$d\tilde{s}^2 = \left(1 + c^4\, \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{\mathcal{A}^2}\right) ds^2 \tag{7}$$

or, in terms of the metric tensor,

$$\tilde{g}_{\mu\nu} = \left(1 + c^4 \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{\mathscr{A}^2}\right) g_{\mu\nu} \tag{8}$$

which depends on the squared length of the (spacelike) four-acceleration, $|\ddot{x}|^2 = g_{\sigma\rho}\ddot{x}^\sigma \ddot{x}^\rho$. Particularly interesting is the case in the absence of gravity, $g_{\mu\nu} = \eta_{\mu\nu}$, which corresponds to a flat background. In this case, any accelerating particle experiences a gravitational field given by

$$\tilde{g}_{\mu\nu} = \left(1 + c^4 \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{\mathscr{A}^2}\right) \eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{9}$$

where $h_{\mu\nu} = c^4(\ddot{x}^\sigma \ddot{x}_\sigma/\mathscr{A}^2)\eta_{\mu\nu}$ is the quantum (local) perturbation to the Minkowskian metric. From Eq. (9) it follows that

$$\tilde{g}^{\mu\nu} \sim \left(1 - c^4 \frac{\ddot{x}^\sigma \ddot{x}_\sigma}{\mathscr{A}^2}\right) \eta^{\mu\nu} \tag{10}$$

We stress that this curvature is not induced by matter through the conventional Einstein equation; it is due to the motion in momentum space and vanishes in the limit $\hbar \to 0$. Thus, it represents a quantum correction to the given background geometry, which henceforth, we will assume flat.

3. GENERALIZED UNCERTAINTY PRINCIPLE

Let us now derive the generalized uncertainty principle (2) starting from relation (1), where the tensor metric is induced by the acceleration of a massive particle in a high-energy scattering process.

According to the hypothesis that microscopic space-time should be regarded as a four-dimensional hypersurface locally embedded in the larger eight-dimensional manifold, as discussed in the previous section, accelerated particles can be associated to four-dimensional hypersurfaces whose curvature is, in general, nonvanishing. At this semiclassical level, the effective space-time geometry experienced by interacting particles is curved.

Inserting (9) into (1), one gets

$$[x^\mu, p^\nu] = i\hbar \left(1 + c^4 \frac{(\ddot{x}^\sigma \ddot{x}_\sigma)_m}{\mathscr{A}^2}\right)^{-1} \eta^{\mu\nu} \tag{11}$$

The right-hand side is understood as a c-function. The term $(\ddot{x}^\sigma \ddot{x}_\sigma)_m$ is the mean value of the squared length of the four-acceleration which takes into account the quantum fluctuation of the metric.

Since $\ddot{x}^\mu = (1/mc)\delta p^\mu/\delta s$, δp^μ is the transferred momentum, it follows that

$$(\ddot{x}^\sigma \ddot{x}_\sigma)_m \simeq \frac{1}{m^2c^2\ \delta s^2}\left[\frac{p^i p^j}{|\vec{p}|^2} - \delta^{ij}\right](\delta p^i\ \delta p^j)_m, \qquad i, j = 1, 2, 3 \tag{12}$$

where the high-energy limit $E \gg m$ has been used. Due to the average on the product of transferred momenta, one can assume

$$(\delta p^i\ \delta p^j)_m \sim \Delta p^2 \delta^{ij} \tag{13}$$

Then Eq. (12) reads

$$(\ddot{x}^\sigma \ddot{x}_\sigma)_m \sim -2\,\frac{\Delta p^2}{m^2c^2\ \delta s^2} \tag{14}$$

Δp is the transferred momentum along the x-direction.

As is well known, two noncommutating operators A and B defined in a Hilbert space, for any given state, satisfy the uncertainty relation

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A, B]\rangle\right|$$

If $A = x^\mu$ and $B = p^\nu$, Eqs. (10) and (11) allow us to write

$$\Delta x^\mu \Delta p^\nu \geq \frac{\hbar}{2}\left|\eta^{\mu\nu}\right| \cdot \left|1 - c^4\,\frac{(\ddot{x}^\sigma \ddot{x}_\sigma)_m}{\mathscr{A}^2}\right| \tag{15}$$

From Eq. (14) and for $\mu = \nu = 1$, one obtains

$$\Delta x\ \Delta p \geq \frac{\hbar}{2} + \frac{\hbar c^2}{m^2 \mathscr{A}^2\ \delta s^2}\,\Delta p^2 \tag{16}$$

For $\mathscr{A} = m_P c^3/\hbar$, where $m_P = (\hbar c/G)^{1/2}$, and $\delta s \sim \lambda_c \sim \hbar/mc$, with λ_c the Compton length, this becomes

$$\Delta x\ \Delta p \geq \frac{\hbar}{2} + \frac{\alpha}{c^3}\,G\ \Delta p^2 \tag{17}$$

that is, we recover Eq. (1). α is a free parameter. Equation (17) is the result we want: The geometrical interpretation of QM through a quantization model formulated in a eight-dimensional manifold, implying the existence of an upper limit on the acceleration of particles, leads to the generalized principle of string theory.

It is worthwhile to note that, in the last term of (17), the dependence on $\hbar$ disappears. So this term is not related to quantum fluctuations, but, as the uncertainty principle for strings, is due to the intrinsic extension of particles.

4. CONCLUSIONS

Starting from the uncertainty principle of QM written in a space-time where the effective geometry is induced by the acceleration of particles moving along their worldlines, the generalized uncertainty principle of string theory has been derived.

In this model we have assumed a maximal acceleration as a universal constant expressed in term of the Planck mass, whose value is $\mathscr{A} \sim 10^{52}$ m/sec^2. As expected, it becomes relevant at very high energy where the emergence of a minimal observable distance occurs.

Unlike string theory, in which the extension of particles is introduced *ab initio*, in quantum geometry such an extension is taken into account through the constraint of the maximal acceleration, that is, by modifying the geometry in which an accelerating particle moves.

In this sense, we can state that the geometrical formulation of QM is an alternative approach in order to study the physics of extended objects.

However, we have to note that we have not used any second quantization scheme or full QFT approach in deriving our generalized uncertainty principle; nevertheless it is indicative of the fact that quantum geometry is an alternative scheme leading to physically interesting results.

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